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Investigation and comparison of sampling properties of L-moments and conventional moments

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Abstract

The first part of this article deals with fitting of regression equations for the sampling properties, variance of L-standard deviation (l_2), and bias and variance of L-skewness (t_3), based on Monte-Carlo simulation results, for generalised Normal (Lognormal-3) and Pearson-3 distributions. These fitted equations will be useful in formulating goodness-of-fit test statistics in regional frequency analysis. The second part presents a comparison of the sampling properties between L-moments and conventional product moments for generalised Normal, generalised Extreme Value, generalised Pareto and Pearson-3 distributions, in a *relative form*. The comparison reveals that the bias in L-skewness is found to be insignificant up to a skewness of about 1.0, even for small samples. In case of higher skewness, for a reasonable sample size of 30, L-skewness is found to be nearly unbiased. However, the conventional skewness is found to be significantly biased, even for a low skewness of 0.5 and a reasonable sample size of 30. The overall performance evaluation in terms of "Relative-RMSE in third moment ratio" reveals that *conventional moments are preferable at lower skewness, particularly for smaller samples, while L-moments are preferable at higher skewness, for all sample sizes*. This point is illustrated through an application that seeks to obtain an appropriate regional flood frequency distribution for the 98 catchment areas located in the central region of India, spread over six hydrometeorologic subzones.

Keywords: Sampling methods; Generalised Normal, Pearson distributions; L-moments; L-skewness

1. Introduction

The method of moments has been one of the simplest and conventional parameter estimation techniques used in statistical hydrology. In this method, while fitting a probability distribution to a sample, the parameters are estimated by equating the sample moments to those of the theoretical moments of the distribution. Even though this method is conceptually simple, and the computations are straight-forward, it is found that the numerical values of the sample moments can be very different from those of the population from which the sample has been drawn, especially when the sample size is small and/or the skewness of the sample is considerable. If μ, σ^2, γ, denote the mean, the variance and the skewness of the population, then the unbiased sample estimates of μ, σ^2, γ, are expressed as:

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{1a}$$

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$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - m)^2, \tag{1b}$$

$$g = \frac{1}{s^3}\left[\frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(x_i - m)^3\right]. \tag{1c}$$

It is to be noted that Eq. 1(c) yields unbiased estimate of skewness, only if the distribution is normal. The attempts made to develop unbiased estimates of skewness for various non-normal distributions usually result in a considerable increase in the variance of skewness (Vogel and Fennessey, 1993). Moreover, a notable drawback with conventional moment ratios such as skewness and coefficient of variation, is that, for finite samples, they are bounded. These bounds are shown to be dependent only on the sample size, irrespective of the population values (Kirby, 1974). Wallis et al., 1974 have shown that the sample estimates of skewness are highly biased for small samples. Recently, Vogel and Fennessey (1993) have investigated the sampling properties of conventional moment ratios for highly skewed Lognormal (LN-2) and generalised Pareto (GPA) distributions for large samples, in terms of bias and root mean square error (RMSE) expressed in a relative form. Their study reveals that sample estimates of coefficient of variation and skewness are remarkably biased even for extremely large samples ($n > 1000$) drawn from highly skewed populations.

2. L-moments

Recently, Hosking (1990) has defined L-moments, which are analogous to conventional moments, and can be expressed in terms of linear combinations of order statistics. Basically, L-moments are linear functions of probability weighted moments (PWMs). Greenwood et al. (1979) defined PWMs as

$$M_{p,r,s} = E(x^p.[F(x)]^r[1 - F(x)]^s) \tag{2}$$

where $F(x)$ is the cumulative distribution function of x, with p,r,s being real numbers.

This expression for $M_{p,r,s}$ (Eq. (2)) is useful in characterising the statistical properties of an observed data set. The quantities $\{M_{p,0,0},\ p = 1,2,3,\ldots\}$ represent the conventional non-central moments of x. But, in case of PWMs, the moments of a distribution are estimated with $M_{1,r,s}$ considering x as linear. In fact, either the quantity $M_{1,0,s}$ (α_s: $s = 0,1,\ldots$) or $M_{1,r,0}$ (β_r: $r = 0,1,\ldots$) is sufficient to characterise a probability distribution. The quantity $M_{1,r,0}$ which is known as β_r, is more commonly used and can be defined as:

$$\beta_r = E(x(F(x))^r) \tag{3}$$

The unbiased sample estimator (b_r) of β_r can be estimated using the expression given by Landwehr et al. (1979):

$$b_r = \frac{1}{n}\frac{\sum_{i=1}^{n}\binom{i-1}{r}x_{(i)}}{\binom{n-1}{r}} \qquad r = 0,1,\ldots,(n-1) \tag{4}$$

where $x_{(i)}$ is an ordered set of observations $x_{(1)} \le x_{(2)} \le x_{(3)} \le \cdots \le x_{(n)}$.

The first four L-moments are given as:

$$\begin{aligned}\lambda_1 &= \beta_0\\ \lambda_2 &= 2\beta_1 - \beta_0\\ \lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0\\ \lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0\end{aligned} \tag{5}$$

The sample estimates (l_r: $r = 1,2,3,\ldots$) of L-moments ($\lambda_1, \lambda_2,\ldots, \lambda_r$) can be obtained by substituting the unbiased sample estimates of β_r given by Eq. (4) into Eq. (5).

More recently, Wang (1996) has derived direct estimators of L-moments by closely following the basic definition of L-moments which is based on order statistics. This eliminates the need to resort to PWMs, while defining the L-moments. However, the numerical values of the sample estimates from the PWM-based indirect estimators and the order statistics based direct estimators are the same.

3. L-moment ratios

Analogous to the conventional statistical moment ratios, the L-moment ratios can be expressed as:

$$\text{L-coefficient of variation}(\tau_2) = \lambda_2/\lambda_1 \tag{6a}$$

$$\text{L-skewness}(\tau_3) = \lambda_3/\lambda_2 \tag{6b}$$

$$\text{L-kurtosis}(\tau_4) = \lambda_4/\lambda_2 \tag{6c}$$

In general,

$$\tau_r = \lambda_r/\lambda_2 \qquad r = 3, 4, \ldots \tag{7}$$

The sample estimates of L-moment ratios $(t_2, t_3, \ldots, t_r)$ can be obtained by replacing $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ by their respective sample estimates $l_1, l_2, \ldots, l_r$ in the previous set of Eqs. (6a)–(6c) and (7).

L-moments are capable of characterising a wider range of distributions, compared to the conventional moments. A distribution may be specified by its L-moments, even if some of its conventional moments do not exist (Hosking, 1990). For example, in the case of the generalised Pareto distribution, the conventional skewness is undefined beyond a value of 155 (shape parameter = − 1/3), while the L-skewness can be defined, even beyond that value. Further, L-moments are more robust to outliers in data than conventional moments (Vogel and Fennessey, 1993) and enable more reliable inferences to be made from small samples about an underlying probability distribution. The advantages offered by L-moments over conventional moments in hypothesis testing, boundedness of moment ratios and identification of distributions have been discussed in detail by Hosking (1986; 1990). Stedinger et al. (1993) have described the theoretical properties of the various distributions commonly used in hydrology, and have summarised the relationships between the distribution parameters and the L-moments.

Hosking (1990) has also introduced L-moment ratio diagrams, which are quite useful in selecting appropriate regional frequency distributions of hydrologic and meteorological data. The advantages offered by L-moment ratio diagram over conventional moment ratio diagram are well elucidated by Vogel and Fennessey (1993). Examples for the use of L-moment ratio diagram are found in the works ofWallis (1988; 1989); Hosking and Wallis (1987; 1993; 1997); Vogel et al. (1993a, b). Of late, many regional frequency studies in Hydrology use the L-moment ratio diagram to select the most appropriate distribution.

Exact variances of the sample PWM's (a_r and b_r) and L-moments can be obtained in terms of expectations of quadratic functions of order statistics,

$$\underset{1\le r<s\le t}{\mathrm{E}}(X_{s:t} - X_{r:t})^2 = \binom{n}{t}^{-1} \sum\sum_{1\le j<k\le n} \binom{j-1}{r-1} \times \binom{k-j-1}{s-r-1}\binom{n-k}{t-s}(x_k - x_j)^2, \tag{8}$$

in a distribution free manner. But, these results depend on the joint distributions of order statistics, and hence are extremely complex and the algebra of the same is not easily tractable. Such exact results are available only for the uniform and the exponential distributions. In contrast, the asymptotic distribution theory provides an acceptable alternative to obtain practical, but slightly approximate estimates of variances of sample PWMs and L-moments. If the underlying distribution has finite variance, then, it is possible to demonstrate asymptotic normality and also calculate the asymptotic bias and variance of the sample statistics a_r, b_r, l_r and t_r, using the set of expressions given by Hosking (1986). The asymptotic theory usually provides a good approximation to the exact distribution of sample PWMs and L-moments for samples of size $n \ge 50$ and in most cases adequate approximation even for $n = 20$ (Hosking, 1986). Using asymptotic theory, Hosking (1986) has derived approximate analytical forms for the variance of L-moment and L-moment ratio estimators for probability distributions such as Gumbel, Normal, Logistic, generalised Extreme Value, generalised Logistic, Wakeby. However, for two of the commonly used distributions in Hydrology, namely, generalised Normal (GNO, more popularly known as log normal 3-parameter distribution) and Pearson-3 (P-3, commonly known as Gamma distribution), neither exact nor approximate analytical estimates are available (Hosking, 1986). Hence, in this article, it is envisaged to fit regression equations for the sampling variances of l_2 and t_3 for these two distributions based on the results obtained from Monte-Carlo simulations. These equations will be useful in formulating homogeneity and goodness of fit (GOF) measures in Regional Frequency Analysis.

In recent literature (Hosking, 1990), it is stated that L-moments in general, are less subject to bias in estimation compared to conventional moment estimators.

Table 1
L-skewness-skewness-L-standard deviation relationships for the four distributions considered in the study with Cv = 1.0

Distribution	L-skewness	Skewness	L-standard deviation
GNO	0.040	0.243	0.5632
	0.080	0.498	0.5585
	0.120	0.766	0.5532
	0.160	1.054	0.5440
	0.220	1.550	0.5260
	0.270	2.055	0.5065
	0.310	2.551	0.4885
	0.340	3.000	0.4732
	0.390	3.963	0.4445
	0.430	5.025	0.4193
GEV	0.050	0.268	0.5648
	0.090	0.520	0.5597
	0.120	0.729	0.5540
	0.160	1.030	0.5460
	0.210	1.557	0.5262
	0.240	1.974	0.5120
	0.270	2.529	0.4959
	0.290	3.035	0.4835
	0.320	4.189	0.4625
	0.330	4.766	0.4546
GPA	0.070	0.256	0.5750
	0.130	0.507	0.5677
	0.180	0.750	0.5581
	0.230	1.045	0.5445
	0.290	1.514	0.5217
	0.340	2.093	0.4962
	0.360	2.415	0.4840
	0.390	3.081	0.4629
	0.420	4.180	0.4381
	0.430	4.736	0.4290
P-3	0.090	0.550	0.5588
	0.170	1.031	0.5457
	0.250	1.505	0.5260
	0.340	2.040	0.4977
	0.420	2.536	0.4680
	0.490	3.007	0.4391
	0.620	4.061	0.3780
	0.710	5.056	0.3290
	0.770	5.962	0.2921

Further, based on limited sampling experiments conducted for a 2-parameter Gumbel population, Wallis (1989) has stated that “even with n small, the τ_r tend to be unbiased, to have small variance and to be normally distributed about their population values”. However, a detailed comparison of the sampling properties between conventional moments and L-moments for various distributions has not been attempted so far. The results of such a comparison, will be useful in statistical hydrology applications in choosing the appropriate moment estimator between conventional moments and L-moments depending on the distribution considered, the level of skewness and the available record length.

4. Present study

This article consists of two distinct parts. The first part addresses the fitting of generalised regression relationships for: (i) variance of the sample estimator of L-standard deviation (l_2) and (ii) bias and variance of the sample estimator of L-skewness (t_3) based on Monte-Carlo simulations for two of the 3-parameter probability distributions often used in hydrologic studies, namely, generalised Normal (GNO, more popularly known as LN-3) and Pearson-3 (P-3). The second part presents a comparison of the sampling properties, root mean square error of second moment and bias and root mean square error of third moment ratio, between L-moments and conventional product moments, in a relative form, for four popular distributions used in statistical hydrology. The four distributions considered are: generalised Normal (GNO), generalised Extreme Value (GEV), generalised Pareto (GPA) and Pearson-3 (P-3). The results of this study would help in a better understanding and interpretation of moment ratio diagrams in the context of the selection of regional frequency distributions for hydrologic studies. This is illustrated through an application towards the end of this article.

5. Experimental design

For the evaluation of the sampling properties, namely, RMSE of l_2; and bias and RMSE of t_3 and to compare the same with RMSE of ‘s’ and bias and RMSE of ‘g’, a detailed Monte-Carlo simulation is carried out, in this study. The details of the experimental design for the same are presented below.

For the conventional method of moments (CMOM), the following population statistics are assumed for generation: μ (mean) = 1; σ (standard deviation) = 1; and γ (skewness) equal to the values given in Table 1. For L-moments (LMOM), the following equivalent values of population statistics are assumed for generation: l_1 (L-mean) = 1; l_2

Table 2
Generating algorithm and parameters-moments relationships

Generating algorithm[a]	Conventional moments	L-moments
Log normal-3		
$x = c + \exp(a + b\phi^{-1}(F))$	$\mu = c + \exp(a + b^2/2)$	$k \cong s(0.999281 - 0.006118s^2 + 0.000127s^4)$ wherein $s = -\sqrt{8/3}\phi^{-1}((1 + \tau_3)/2)$
c = location parameter	$\sigma^2 = \exp^2(a)[\exp(b^2)*(\exp(b^2\text{-}1))]$	$\lambda_2 = \alpha(k e^{-k^2/2})/(1\text{-}2\phi(-k/\sqrt{2}))$
a = mean of the log values	$\gamma = \dfrac{(\exp(3b^2) - 3\exp(b^2) + 2)}{\{\exp(b^2) - 1\}^{3/2}}$	$\lambda_1 = \xi + \alpha k^{-1}(1 - e^{-k^2/2})$
b = standard deviation of log values		$k = -b,\ \ \alpha = b e^a,\ \ \xi = c + e^a$
GEV		
$x = \xi + (\alpha/k)(1 - (-\ln(F))^k)$	$\mu = \xi + \alpha/k(1 - \Gamma(1 + k))$	$\kappa = 7.8590c + 2.9554c^2$
ξ = location parameter		$c = \dfrac{\{(2\beta_1 - \beta_0)\}}{\{(3\beta_2 - \beta_0)\}} - \dfrac{\{\log 2\}}{\{\log 3\}}$
α = scale parameter	$\sigma^2 = (\alpha/k)^2(\Gamma(1 + 2k) - \Gamma^2(1 + k))$	$\lambda_2 = \alpha(1 - 2^{-k})\Gamma(1 + k)/k$
k = shape parameter	$\gamma = \dfrac{(\Gamma(1 + 3k) - 3\Gamma(1 + 2k)\Gamma(1 + k) + 2\Gamma^3(1 + k))}{(\Gamma(1 + 2k) - \Gamma^2(1 + k))^{3/2}}$	$\lambda_1 = \xi + (\alpha/k)(1 - \Gamma(1 + k))$
GPA		
$x = \xi + (\alpha/k)(1 - (1 - F)^k)$	$\mu = \xi + \alpha/(1 + k)$	$\lambda_1 = \xi + \alpha/(1 + k)$
ξ = location parameter	$\sigma^2 = \alpha^2/((1 + k^2)(1 + 2k))$	$\lambda_2 = \alpha/((1 + k)(2 + k))$
α = scale parameter	$\gamma = 2(1 - k)(1 + k)^{1/2}/(1 + 3k)$	$\tau_3 = (1 - k)/(3 + k)$
k = shape parameter		
Pearson-3		
$x = \mu + \sigma K(\gamma, F)$	$\mu = \xi + \alpha k$	$\lambda_1 = \xi + \alpha k$
$K(\gamma,F)$ = Frequency factor	$\sigma = \alpha k^{1/2}$	$\lambda_2 = \pi^{-1/2}\alpha\Gamma(k + 0.5)/\Gamma(k)$
ξ = *location* parameter	$\gamma = 2/k^{1/2}$	$\tau_3 = 6I_{1/3}(k, 2k) - 3$
α = *scale* parameter		
k = *shape* parameter		

[a] Here $\phi^{-1}(F)$: Inverse of the standard normal function. $I_x(p,q) = \dfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^x t^{p-1}(1-t)^{q-1}dt = \beta$-function ratio (can be computed using Fortran routines of Hosking (1991)).

(L-standard deviation) and τ_3 (L-skewness) values are given in Table 1. For each of the four distributions considered, for the assumed values of conventional skewness, the corresponding L-skewness values (given in Table 1), have been taken from the conversion table of Hosking (Personal Communication, 1995). Further, for the assumed standard deviation (σ), the equivalent L-standard deviation (λ_2) has been obtained analytically, using the respective parameters-moments relationships given in Table 2. Thus, the values of l_2 given in Table 1 correspond to a conventional Cv of 1.0. Since $\mathrm{Var}(l_2)/\lambda_2^2$, relative RMSE ($l_2$), bias ($t_3$) and Var of ($t_3$) are invariant of Cv (or L-Cv), the Monte-Carlo simulation experiments are conducted for Cv = 1.0 only. For each distribution considered, N = 10 000 samples have been generated for each of the sample sizes, n = 10 (10) 90, and each skewness considered, using the respective generating algorithm given in Table 2. For both conventional moments and L-moments, parameters to be used in the generation are estimated using the relationships between the moments and the

Table 3
Asymptotic normality of L-skewness-distribution: GEV

n	L-skew	$(\bar{t}_3)$[a]	$(\tilde{t}_3)$[b]	$t_3\ (t_3)$[c]	Z[d]
10	0.090	0.083	0.085	− 0.0108	− 0.0664
	0.160	0.143	0.144	− 0.0100	− 0.0614
	0.240	0.208	0.210	− 0.0036	− 0.0220
	0.290	0.247	0.249	0.0011	0.0068
	0.330	0.276	0.276	0.0043	0.0263
30	0.090	0.088	0.088	− 0.0031	− 0.0371
	0.160	0.154	0.154	0.0013	0.0150
	0.240	0.228	0.226	0.0018	0.2149
	0.290	0.273	0.270	0.0313	0.3722
	0.330	0.308	0.302	0.0412	0.4884
90	0.090	0.089	0.088	0.0033	0.0702
	0.160	0.158	0.158	0.0036	0.0766
	0.240	0.236	0.235	0.0170	0.3650
	0.290	0.284	0.282	0.0312	0.6853
	0.330	0.322	0.319	0.0457	0.9805

[a] Here $\bar{t}_3$ = Mean of t_3.
[b] Here $\tilde{t}_3$ = Median of t_3.
[c] Here $t_3\ (t_3)$ = L-skew of t_3 computed from 10000 replicates.
[d] Here $Z = t_3\ (t_3)/(0.1866n^{-1} + 0.8n^{-2})^{0.5}$ based on the Normality test of Hosking (1990). $Z_{crit} = 1.64$ at 10% significance level.

Table 4
Coefficients of the regression equations for variance of L-standard deviation for GNO and P-3 distributions

Distribution	L-skew (τ_3)	Var (l_2)[a]	
		a_1	a_2
GNO	0.040	0.5369	0.5266
	0.080	0.5781	0.5329
	0.120	0.6592	0.4735
	0.160	0.7670	0.4738
	0.220	1.0101	0.4267
	0.270	1.2948	0.4408
	0.310	1.5933	0.4808
	0.340	1.8652	0.5588
	0.390	2.4489	0.7486
	0.430	3.0646	1.0327
P-3	0.090	0.5722	0.5744
	0.120	0.6164	0.5670
	0.170	0.7195	0.5604
	0.250	0.9634	0.5608
	0.340	1.3700	0.5699
	0.420	1.8819	0.5812
	0.490	2.4888	0.5935
	0.620	4.2766	0.6577
	0.710	6.5009	0.8567
	0.770	8.9696	1.2529

[a] Here a_1, a_2 are regression coefficients.

parameters of the distribution (Table 2). The generation of the uniformly distributed pseudo-random numbers has been done using the well-tested subroutine RAN1 from Press et al. (1986).

Since the relationship between conventional skewness and L-skewness is highly non-linear for all the 3-parameter distributions, it is essential to perform the comparison between their sampling properties in the same plane (either in the conventional plane or in the L-plane). It is found that quite a fraction of the 10 000 sample estimates of L-skewness, may overestimate their population value to such an extent that equivalent values of those estimates may not be defined in the conventional plane. Owing to this limitation of the conventional moments, the comparison has been carried out in the L-plane, by converting the sample estimates of conventional skewness (g) to their equivalent values in the L-plane. To maintain uniformity, the comparison between RMSE (s) and RMSE (l_2) is also performed in the L-plane, by converting the sample estimates of conventional standard deviation (s) to their equivalent values in the L-plane.

If $\hat{\theta}$ represents a sample estimate of the population property θ, the mean and the variance of $\hat{\theta}$ for each distribution have been estimated by:

$$\hat{\mu}(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i \qquad (9)$$

$$\hat{\mathrm{Var}}(\hat{\theta}) = \frac{1}{(N-1)}\sum_{i=1}^{N}[\hat{\theta}_i - \hat{\mu}(\hat{\theta})]^2 \qquad (10)$$

where N = 10 000 samples of size n each.

In this study, the sampling properties of LMOM and CMOM are compared based on relative bias (R-BIAS) and relative root mean square error (R-RMSE), which are expressed as:

$$\text{R-BIAS}(\theta) = \frac{1}{\theta}[\theta - \hat{\mu}(\hat{\theta})] \qquad (11)$$

$$\text{R-RMSE}(\theta) = \frac{1}{\theta}[(\theta - \hat{\mu}(\hat{\theta}))^2 + V\hat{a}r(\hat{\theta})]^{1/2} \qquad (12)$$

6. Results and discussion

The sampling distributions of L-moment ratios are known to be asymptotically normal (Hosking, 1990). To start with, this property is verified in this section for the three parameter General Extreme Value (GEV) distribution for a reasonable sample size of 30 and five different skewness values, applying the normality test proposed by Hosking (1990). This normality statistic (Z) is defined as:

$$Z = v_n^{-1/2} t_3(t_3) \quad (13)$$

where $v_n = 0.1866n^{-1} + 0.8n^{-2}$, $t_3(t_3) =$ L-skewness of the sample L-skewnesses estimated from 10 000 samples, $n =$ sample length.

The normality of L-skewness is tested and verified at 10% significance level ($Z_{crit} = 1.64$) and the results are presented in Table 3. This property forms the basis of the formulation of different heterogeneity and goodness of fit measures in regional frequency analysis (for example, Hosking et al., 1985; Chowdhury et al., 1991; Hosking and Wallis, 1993). For this purpose, it is essential to obtain the variance of L-moment ratios, either by means of analytical approximations or through Monte-Carlo simulation. As mentioned earlier, the analytical approximations are not available for generalised normal (GNO) and Pearson-3 (P-3) distributions and hence Monte-Carlo simulations are resorted to in this article. Based on the Monte-Carlo simulation results, regression equations are fitted for variance of sample estimator of L-standard deviation and bias and variance of sample estimator of L-skewness for these two distributions. These fitted equations will be useful in: (i) formulating goodness of fit (GOF) measures based on these distributions, for use in hydrologic applications; and (ii) hydrologic applications in which bias corrections to L-skewness are necessary.

6.1. Regression fitting for variance of L-standard deviation

The asymptotic biases and variances of L-moment statistics are in general of order $1/n$ (Hosking, 1986)

Table 5
Coefficients of the regression equations for bias and variance of L-skewness for GNO and P-3 distributions

		Bias (τ_3)[a]			Var (τ_3)[a]		
Distribution	L-skew (τ_3)	a_0	a_1	a_2	a_0	a_1	a_2
GNO	0.040	0.0003	0.0221	0.0629	0.0001	0.1731	1.1640
	0.080	0.0003	0.0637	0.0805	0.0002	0.1787	1.1459
	0.120	0.0003	0.1083	0.0846	0.0002	0.1887	1.1095
	0.160	0.0003	0.1575	0.0671	0.0002	0.2030	1.0546
	0.220	0.0003	0.2437	−0.0200	0.0003	0.2313	0.9355
	0.270	0.0003	0.3302	−0.1705	0.0004	0.2597	0.8057
	0.310	0.0005	0.4110	−0.3551	0.0005	0.2845	0.6859
	0.340	0.0006	0.4791	−0.5377	0.0006	0.3034	0.5898
	0.390	0.0010	0.6080	−0.9357	0.0009	0.3340	0.4270
	0.430	0.0014	0.7125	−1.3454	0.0011	0.3563	0.3021
P-3	0.090	0.0003	0.0695	0.0913	0.0002	0.1733	1.1578
	0.120	0.0003	0.0962	0.1083	0.0002	0.1756	1.1454
	0.170	0.0002	0.1361	0.1356	0.0002	0.1817	1.1165
	0.250	0.0002	0.1852	0.1724	0.0003	0.1965	1.0615
	0.340	0.0001	0.2178	0.1945	0.0003	0.2169	1.0102
	0.420	0.0001	0.2284	0.1949	0.0003	0.2329	0.9816
	0.490	0.0001	0.2256	0.1864	0.0003	0.2417	0.9609
	0.620	0.0001	0.1951	0.1781	0.0004	0.2374	0.8858
	0.710	0.0002	0.1584	0.1884	0.0003	0.2138	0.7762
	0.770	0.0003	0.1298	0.1859	0.0003	0.1870	0.6730

[a] Here a_0, a_1, a_2 are regression coefficients.

Table 6
Comparison of sampling properties (RMSE of second moment; BIAS and RMSE of third moment ratio in a relative form)-L-moments vs conventional moments-generalised normal distribution

n	Skew	L-skew	r-RMSE (l_2)[a]	r-RMSE (s)[b,g]	r-bias (t_3)[c]	r-bias (g)[d,g]	r-RMSE (t_3)[e]	r-RMSE (g)[f,g]
10	0.498	0.080	0.2518	0.2448	0.0936	0.4538	2.1486	1.3807
10	1.054	0.160	0.2869	0.2836	0.1045	0.4682	1.1065	0.8020
10	2.055	0.270	0.3675	0.3829	0.1174	0.5033	0.6971	0.6294
10	3.000	0.340	0.4393	0.4782	0.1270	0.5285	0.5788	0.6043
10	3.963	0.390	0.5037	0.5684	0.1345	0.5464	0.5212	0.6009
10	5.025	0.430	0.5650	0.6561	0.1409	0.5606	0.4850	0.6032
20	0.498	0.080	0.1742	0.1714	0.0426	0.2910	1.3659	1.0571
20	1.054	0.160	0.1989	0.1998	0.0501	0.3096	0.7133	0.6095
20	2.055	0.270	0.2558	0.2770	0.0595	0.3618	0.4622	0.4802
20	3.000	0.340	0.3078	0.3569	0.0669	0.3987	0.3909	0.4698
20	3.963	0.390	0.3531	0.4334	0.0730	0.4246	0.3563	0.4754
20	5.025	0.430	0.3957	0.5091	0.0784	0.4447	0.3344	0.4840
30	0.498	0.080	0.1421	0.1402	0.0348	0.2242	1.0758	0.9087
30	1.054	0.160	0.1605	0.1617	0.0370	0.2396	0.5655	0.5243
30	2.055	0.270	0.2072	0.2255	0.0428	0.2968	0.3715	0.4148
30	3.000	0.340	0.2486	0.2924	0.0483	0.3372	0.3175	0.4089
30	3.963	0.390	0.2847	0.3582	0.0530	0.3658	0.2915	0.4171
30	5.025	0.430	0.3185	0.4242	0.0574	0.3880	0.2751	0.4277
60	0.498	0.080	0.0939	0.0928	0.0192	0.1352	0.7408	0.6978
60	1.054	0.160	0.0977	0.0968	0.0194	0.1463	0.3928	0.4098
60	2.055	0.270	0.1120	0.1131	0.0225	0.2052	0.2626	0.3286
60	3.000	0.340	0.1459	0.1591	0.0257	0.2480	0.2275	0.3256
60	3.963	0.390	0.1758	0.2099	0.0287	0.2789	0.2110	0.3348
60	5.025	0.430	0.2020	0.2629	0.0315	0.3032	0.2008	0.3464
90	0.498	0.080	0.0796	0.0788	0.0134	0.0956	0.5937	0.5888
90	1.054	0.160	0.0913	0.0921	0.0131	0.1054	0.3165	0.3534
90	2.055	0.270	0.1186	0.1286	0.0150	0.1612	0.2136	0.2874
90	3.000	0.340	0.1428	0.1707	0.0172	0.2033	0.1863	0.2846
90	3.963	0.390	0.1640	0.2167	0.0194	0.2344	0.1738	0.2933
90	5.025	0.430	0.1839	0.2662	0.0215	0.2592	0.1662	0.3047

[a] Here r-RMSE (l_2)-Relative RMSE in L-standard deviation.
[b] Here r-RMSE (s)-Relative RMSE in standard deviation.
[c] Here r-bias (t_3)-Relative bias in L-skewness.
[d] Here r-bias (g)-Relative bias in skewness.
[e] Here r-RMSE (t_3)-relative RMSE in L-skewness.
[f] Here r-RMSE (g)-Relative RMSE in skewness.
[g] Converted to equivalent values in the L-plane.

and ($\mathrm{Var}(l_2)/\lambda_2^2$) is invariant of Cv (or L-Cv). Hence, the general form of the equation for $\mathrm{Var}(l_2)$ corresponding to a given L-skewness is given by:

$$\mathrm{Var}(l_2) = \lambda_2^2(a_1 n^{-1} + a_2 n^{-2}). \qquad (14)$$

The coefficients a_1 and a_2 for the regression fitting of $\mathrm{Var}(l_2)$ are presented in Table 4, for the two distributions, for various L-skewness values.

6.2. Regression fitting for bias and variance of L-skewness

As the bias and the variance of L-skewness are invariant of L-Cv, the regression equations fitted for the bias and the variance of L-skewness are of the general form:

$$V = a_0 + a_1 n^{-1} + a_2 n^{-2} \qquad (15)$$

Table 7
Comparison of sampling properties (RMSE of second moment; BIAS and RMSE of third moment ratio in a relative form)-L-moments vs conventional moments-generalised Extreme value distribution

n	Skew	L-skew	r-RMSE (l_2)[a]	r-RMSE (s)[b,g]	r-bias (t_3)[c]	r-bias (g)[d,g]	r-RMSE (t_3)[e]	r-RMSE (g)[f,g]
10	0.520	0.090	0.2484	0.2410	0.0788	0.4468	1.8866	1.2541
10	1.030	0.160	0.2865	0.2811	0.1053	0.4785	1.1137	0.8024
10	1.974	0.240	0.3565	0.3727	0.1321	0.5130	0.7964	0.6594
10	3.035	0.290	0.4182	0.4599	0.1497	0.5392	0.6915	0.6343
10	4.189	0.320	0.4641	0.5278	0.1605	0.5547	0.6454	0.6294
10	4.766	0.330	0.4812	0.5537	0.1642	0.5597	0.6321	0.6289
20	0.520	0.090	0.1715	0.1678	0.0342	0.2570	1.1893	0.9013
20	1.030	0.160	0.1995	0.1987	0.0501	0.3102	0.7180	0.5781
20	1.974	0.240	0.2497	0.2739	0.0680	0.3643	0.5321	0.4852
20	3.035	0.290	0.2935	0.3509	0.0807	0.4034	0.4730	0.4797
20	4.189	0.320	0.3258	0.4131	0.0891	0.4261	0.4475	0.4848
20	4.766	0.330	0.3379	0.4375	0.0921	0.4335	0.4401	0.4875
30	0.520	0.090	0.1395	0.1368	0.0276	0.1853	0.9335	0.7467
30	1.030	0.160	0.1620	0.1617	0.0370	0.2403	0.5687	0.4836
30	1.974	0.240	0.2022	0.2241	0.0494	0.2983	0.4287	0.4105
30	3.035	0.290	0.2368	0.2907	0.0591	0.3417	0.3861	0.4118
30	4.189	0.320	0.2622	0.3457	0.0658	0.3672	0.3681	0.3672
30	4.766	0.330	0.2716	0.3671	0.0681	0.3755	0.3630	0.4244
60	0.520	0.090	0.0965	0.0949	0.0151	0.1004	0.6402	0.5432
60	1.030	0.160	0.1129	0.1129	0.0194	0.1504	0.3946	0.3644
60	1.974	0.240	0.1421	0.1596	0.0262	0.2076	0.3041	0.3146
60	3.035	0.290	0.1675	0.2158	0.0320	0.2552	0.2790	0.3213
60	4.189	0.320	0.1862	0.2658	0.0362	0.2839	0.2693	0.3334
60	4.766	0.330	0.1931	0.2860	0.0378	0.2934	0.2666	0.3383
90	0.520	0.090	0.0788	0.0774	0.0106	0.0671	0.5122	0.4439
90	1.030	0.160	0.0920	0.0918	0.0131	0.1116	0.3178	0.3071
90	1.974	0.240	0.1155	0.1300	0.0175	0.1646	0.2477	0.2688
90	3.035	0.290	0.1361	0.1808	0.0216	0.2128	0.2295	0.2769
90	4.189	0.320	0.1515	0.2295	0.0246	0.2426	0.2231	0.2426
90	4.766	0.330	0.1572	0.2498	0.0257	0.2525	0.2215	0.2954

[a] Here r-RMSE (l_2)-Relative RMSE in L-standard deviation.
[b] Here r-RMSE (s)-Relative RMSE in standard deviation.
[c] Here r-bias (t_3)-Relative bias in L-skewness.
[d] Here r-bias (g)-Relative bias in skewness.
[e] Here r-RMSE (t_3)-relative RMSE in L-skewness.
[f] Here r-RMSE (g)-Relative RMSE in skewness.
[g] Converted to equivalent values in the L-plane.

where V is bias or variance in L-skewness, n is the sample length and a_0, a_1, and a_2 are coefficients of the regression fitting. The coefficients a_0, a_1, and a_2 are presented in Table 5, for the two distributions and the respective L-skewness values considered in the study. The regression equations fitted for var (l_2), bias (t_3) and var (t_3) are found to reproduce the simulated values (exact to three decimal places).

7. Comparison of sampling properties—L-moments versus conventional moments

The comparison of results between conventional moments and L-moments for the sampling properties namely, root mean square error of second moment; and bias and root mean square error of third moment ratio, is presented in a *relative form* in the L-plane in Tables 6–9, for the four probability distributions

Table 8
Comparison of sampling properties (RMSE of second moment; BIAS and RMSE of third moment ratio in a relative form)-L-moments versus conventional moments-generalised Pareto distribution

n	Skew	L-skew	r-RMSE (l_2)[a]	r-RMSE (s)[b,g]	r-bias(t_3)[c]	r-bias (g)[d,g]	r-RMSE (t_3)[e]	r-RMSE (g)[f,g]
10	0.507	0.130	0.2053	0.2023	− 0.0067	0.2231	1.2203	1.0276
10	1.045	0.230	0.2685	0.2632	0.0288	0.2637	0.7175	0.5822
10	2.093	0.340	0.3816	0.3900	0.0741	0.3478	0.5341	0.4771
10	3.081	0.390	0.4574	0.4895	0.0978	0.3871	0.4911	0.4753
10	4.180	0.420	0.5149	0.5708	0.1127	0.4097	0.4715	0.4806
10	4.736	0.430	0.5368	0.6022	0.1178	0.4171	0.4657	0.4832
20	0.507	0.130	0.1384	0.1372	− 0.0067	0.0853	0.7540	0.6506
20	1.045	0.230	0.1858	0.1846	0.0099	0.1289	0.4457	0.3683
20	2.093	0.340	0.2678	0.2824	0.0354	0.2207	0.3469	0.3175
20	3.081	0.390	0.3216	0.3667	0.0507	0.2669	0.3284	0.3304
20	4.180	0.420	0.3621	0.4393	0.0612	0.2941	0.3185	0.3488
20	4.736	0.430	0.3774	0.4681	0.0649	0.3031	0.3185	0.3488
30	0.507	0.130	0.1119	0.1113	− 0.0039	0.0511	0.5941	0.5177
30	1.045	0.230	0.1509	0.1508	0.0069	0.0859	0.3505	0.2931
30	2.093	0.340	0.2172	0.2303	0.0249	0.1707	0.2773	0.2581
30	3.081	0.390	0.2601	0.3019	0.0364	0.2172	0.2662	0.2739
30	4.180	0.420	0.2920	0.3655	0.0446	0.2454	0.2624	0.2890
30	4.736	0.430	0.3040	0.3910	0.0475	0.2548	0.2614	0.2948
60	0.507	0.130	0.0769	0.0766	− 0.0025	0.0222	0.4064	0.3548
60	1.045	0.230	0.1052	0.1059	0.0031	0.0425	0.2399	0.2034
60	2.093	0.340	0.1530	0.1629	0.0126	0.1082	0.1937	0.1869
60	3.081	0.390	0.1841	0.2190	0.0192	0.1521	0.1898	0.2031
60	4.180	0.420	0.2074	0.2743	0.0242	0.1805	0.1900	0.2189
60	4.736	0.430	0.2162	0.2975	0.0260	0.1901	0.1903	0.2250
90	0.507	0.130	0.0628	0.0626	− 0.0008	0.0142	0.3236	0.2833
90	1.045	0.230	0.0858	0.0866	0.0023	0.0277	0.1918	0.1637
90	2.093	0.340	0.1244	0.1323	0.0084	0.0809	0.1566	0.1556
90	3.081	0.390	0.1494	0.1799	0.0129	0.1220	0.1551	0.1707
90	4.180	0.420	0.1685	0.2313	0.0164	0.1497	0.1566	0.1860
90	4.736	0.430	0.1757	0.2540	0.0177	0.1593	0.1573	0.1920

[a] Here r-RMSE (l_2)-Relative RMSE in L-standard deviation.
[b] Here r-RMSE (s)-Relative RMSE in standard deviation.
[c] Here r-bias (t_3)-Relative bias in L-skewness.
[d] Here r-bias (g)-Relative bias in skewness.
[e] Here r-RMSE (t_3)-Relative RMSE in L-skewness.
[f] Here r-RMSE (g)-Relative RMSE in skewness.
[g] Converted to equivalent values in the L-plane.

considered in this article. *It is to be noted that these results are invariant of Cv or L-Cv.*

7.1. Relative-RMSE in second moment

The comparison of results between RMSE of conventional standard deviation and RMSE of L-standard deviation is presented in Tables 6–9. These results are presented for a Cv of 1.0. However, it is to be noted that the relative values of RMSE (s) and RMSE (l_2) (given in columns (5) and (4)) are invariant of Cv (or L-Cv). From Tables 6–9, it is observed that no significant difference in relative RMSE in second moment is observed between conventional moments and L-moments till a skewness of about 3.0. However, as the skewness increases, L-moments are found to outperform conventional moments. For all the four distributions considered, this trend of behaviour in relative RMSE of second moment is found to be the same at all sample lengths.

Table 9
Comparison of sampling properties (RMSE of second moment; BIAS and RMSE of third moment ratio in a relative form)-L-moments vs conventional moments-Pearson-3 distribution

n	Skew	L-skew	r-RMSE (l_2)[a]	r-RMSE (s)[b,g]	r-bias(t_3)[c]	r-bias (g)[d,g]	r-RMSE (t_3)[e]	r-RMSE (g)[f,g]
10	0.550	0.090	0.2509	0.2439	0.0907	0.3755	1.8969	1.3405
10	1.031	0.170	0.2784	0.2730	0.0894	0.3897	1.0152	0.7946
10	2.040	0.340	0.3777	0.3825	0.0702	0.4279	0.5315	0.5588
10	3.007	0.490	0.5047	0.5347	0.0501	0.4620	0.3804	0.5256
10	4.061	0.620	0.6588	0.7234	0.0350	0.4840	0.2950	0.5210
10	5.056	0.710	0.8113	0.9250	0.0253	0.4945	0.2432	0.5200
20	0.550	0.090	0.1733	0.1704	0.0413	0.2202	1.2023	1.0049
20	1.031	0.170	0.1935	0.1924	0.0418	0.2397	0.6470	0.6137
20	2.040	0.340	0.2650	0.2735	0.0331	0.2950	0.3442	0.4426
20	3.007	0.490	0.3556	0.3850	0.0237	0.3397	0.2485	0.4186
20	4.061	0.620	0.4652	0.5347	0.0120	0.3780	0.1600	0.4130
20	5.056	0.710	0.5734	0.6948	0.0117	0.3778	0.1599	0.4131
30	0.550	0.090	0.1409	0.1390	0.0330	0.1609	0.9461	0.8567
30	1.031	0.170	0.1571	0.1568	0.0306	0.1783	0.5109	0.5391
30	2.040	0.340	0.2150	0.2222	0.0232	0.2337	0.2747	0.3939
30	3.007	0.490	0.2885	0.3129	0.0165	0.2791	0.1998	0.3691
30	4.061	0.620	0.3773	0.4363	0.0110	0.3060	0.1560	0.3630
30	5.056	0.710	0.4645	0.5693	0.0083	0.3171	0.1294	0.3592
60	0.550	0.090	0.0975	0.0968	0.0180	0.0886	0.6510	0.6457
60	1.031	0.170	0.1094	0.1099	0.0158	0.1013	0.2491	0.4307
60	2.040	0.340	0.1514	0.1570	0.0117	0.1490	0.1915	0.3271
60	3.007	0.490	0.2042	0.2217	0.0084	0.1912	0.1399	0.2996
60	4.061	0.620	0.2678	0.3105	0.0060	0.2170	0.1096	0.2880
60	5.056	0.710	0.3302	0.4072	0.0044	0.2270	0.0908	0.2790
90	0.550	0.090	0.0796	0.0789	0.0125	0.0590	0.5217	0.5416
90	1.031	0.170	0.0892	0.0895	0.0107	0.0694	0.2841	0.3738
90	2.040	0.340	0.1231	0.1281	0.0078	0.1098	0.1547	0.2930
90	3.007	0.490	0.1659	0.1811	0.0056	0.1479	0.1132	0.2650
90	4.061	0.620	0.2173	0.2534	0.0040	0.1720	0.0890	0.2500
90	5.056	0.710	0.2678	0.3319	0.0029	0.1815	0.0739	0.2388

[a] Here r-bias (t_3)-Relative bias in L-skewness.
[b] Here r-RMSE (s)-Relative RMSE in standard deviation.
[c] Here r-bias (t_3)-Relative bias in L-skewness.
[d] Here r-bias (g)-Relative bias in skewness.
[e] Here r-RMSE (t_3)-Relative RMSE in L-skewness.
[f] Here r-RMSE (g)-Relative RMSE in skewness.
[g] Converted to equivalent values in the L-plane.

7.2. Relative-bias in third moment ratio

The relative bias in conventional skewness is quite significant, even for a low skewness of 0.5, when the sample size is less than 60. Moreover, the relative bias in conventional skewness increases phenomenally with increase in the population skewness and at higher skewness, even for a sample length of 90, the relative bias is seen to be considerable. In fact, even for a population skewness of 1.0, a sample size of about 90 is required, to obtain a reasonably unbiased estimate of skewness. On the contrary, the relative bias in L-skewness is found to be very small upto a skewness of 1.0, even for a sample size of 20. Further, for a reasonable sample size of 30, for high skewness values, the relative bias in L-skewness is seen to be very small. A consistent observation from Tables 6–9 is that at higher values of skewness, the relative bias in

Table 10
Summarised table of comparison of sampling properties of third moment ratio for four distributions-L-moments vs conventional moments

Dist	n	skew	L-skew	r-bias (t_3)[a]	r-bias (g)[b,g]	r-std (t_3)[c]	r-std (g)[d,g]	r-RMSE (t_3)[e]	r-RMSE (g)[f,g]
GNO	10	1.054	0.160	0.1045	0.4682	1.013	0.6513	1.1065	0.8020
		3.000	0.340	0.1270	0.5285	0.5647	0.2932	0.5788	0.6043
		5.025	0.430	0.1409	0.5606	0.4642	0.2225	0.4850	0.6032
	30	1.054	0.160	0.0370	0.2396	0.5644	0.4663	0.5655	0.5243
		3.000	0.340	0.0483	0.3372	0.3138	0.2312	0.3175	0.4089
		5.025	0.430	0.0574	0.3880	0.2691	0.1800	0.2751	0.4277
	90	1.054	0.160	0.0131	0.1054	0.3163	0.3375	0.3165	0.3534
		3.000	0.340	0.0172	0.2033	0.1855	0.1991	0.1863	0.2846
		5.025	0.430	0.0215	0.2592	0.1648	0.1602	0.1662	0.3047
GEV	10	1.030	0.160	0.1053	0.4785	1.1087	0.6441	1.1137	0.8024
		3.035	0.290	0.1497	0.5392	0.6751	0.3339	0.6915	0.6343
		4.766	0.330	0.1642	0.5597	0.6104	0.2868	0.6321	0.6289
	30	1.030	0.160	0.0370	0.2403	0.5675	0.4197	0.5687	0.4836
		3.035	0.290	0.0591	0.3417	0.3815	0.2297	0.3861	0.4118
		4.766	0.330	0.0681	0.3755	0.3566	0.1976	0.3630	0.4244
	90	1.030	0.160	0.0131	0.1116	0.3175	0.2861	0.3178	0.3071
		3.035	0.290	0.0216	0.2128	0.2285	0 1773	0.2295	0.2769
		4.766	0.330	0.0257	0.2525	0.2200	0.1532	0.2215	0.2954
GPA	10	1.045	0.230	0.0288	0.2637	0.7169	0.5190	0.7175	0.5822
		3.081	0.390	0.0978	0.3871	0.4813	0.2759	0.4911	0.4753
		4.736	0.430	0.1178	0.4171	0.4506	0.2439	0.4657	0.4832
	30	1.045	0.230	0.0069	0.0859	0.3504	0.2802	0.3505	0.2931
		3.081	0.390	0.0364	0.2172	0.2637	0.1668	0.2662	0.2739
		4.736	0.430	0.0475	0.2548	0.2571	0.1483	0.2614	0.2948
	90	1.045	0.230	0.0023	0.0277	0.1917	0.1614	0.1918	0.1637
		3.081	0.390	0.0129	0.1220	0.1546	0.1194	0.1551	0.1707
		4.736	0.430	0.0177	0.1593	0.1543	0.1072	0.1573	0.1920
P-3	10	1.031	0.170	0.0894	0.3897	1.0112	0.6924	1.0152	0.7946
		3.007	0.490	0.0501	0.4620	0.3771	0.2508	0.3804	0.5266
		5.056	0.710	0.0253	0.4945	0.2418	0.1608	0.2432	0.5200
	30	1.031	0.170	0.0306	0.1783	0.5112	0.5088	0.5109	0.5391
		3.007	0.490	0.0165	0.2791	0.1992	0.2416	0.1998	0.3691
		5.056	0.710	0.0083	0.3171	0.1292	0.1687	0.1294	0.3592
	90	1.031	0.170	0.0107	0.0694	0.2841	0.3671	0.2841	0.3738
		3.007	0.490	0.0056	0.1479	0.1131	0.2200	0.1132	0.2650
		5.056	0.710	0.0029	0.1815	0.0738	0.1552	0.0739	0.2388

[a] Here r-bias (t_3)-Relative bias in L-skewness.
[b] Here r-bias (g)-Relative bias in skewness.
[c] Here r-std (t_3)-Relative standard deviation in L-skewness.
[d] Here r-std (g)-Relative standard deviation in skewness.
[e] Here r-RMSE (t_3)-Relative RMSE in L-skewness.
[f] Here r-RMSE (g)-Relative RMSE in skewness.
[g] Converted to equivalent values in the L-plane.

conventional skewness for a sample size of 90, is found to be significantly higher compared to the relative bias in L-skewness for a small sample of size 10. Thus, L-moments are found to be far less biased in the third moment ratio, compared to the conventional moments.

7.3. Relative-RMSE in third moment ratio

It is observed from Tables 6–9 that the overall performance of conventional moments is found to be better for lower skewness values, while the L-moments perform better at higher skewness.

However, it is to be noted that even for lower skewness values, with increase in sample size, the performance of L-moments improves considerably and the difference in relative RMSE between conventional moments and L-moments reduces phenomenally. On the contrary, at higher skewness, the performance of conventional moments does not seem to improve considerably with increase in sample size.

A summary of the relative bias (r-bias), the relative standard deviation (r-std) and the relative RMSE (r-RMSE) of L-skewness (t_3) and conventional skewness (g) for a few selected sample lengths and population skewnesses for the four distributions considered, is presented in Table 10. It is observed from Table 10 that for all the four distributions, at lower skewness and short record length, r-std (t_3) is higher than r-std (g). This results in the poorer performance of L-skewness in terms of r-RMSE, even though L-skewness is relatively less biased. However, with increase in sample size: (i) in case of GEV and GPA distributions, it is seen that the difference in r-std of third moment ratio reduces considerably, resulting in only a marginal difference between r-RMSE (t_3) and r-RMSE (g); (ii) for GNO and P-3 distributions, a phenomenal reduction in r-std (t_3) is observed. This results in a lower value of r-std (t_3) compared to r-std (g), in turn, leading to a better performance of L-skewness over conventional skewness in terms of r-RMSE.

Further, it is observed from Table 10 that at higher skewness in case of GEV and GPA distributions, the "difference between r-std (t_3) and r-std (g)" is higher compared to GNO and P-3 distributions, which results in less difference between r-RMSE (t_3) and r-RMSE (g). Another point to be noted is that in the case of GPA distribution, "the difference between r-RMSE (t_3) and r-RMSE (g)" is less compared to that of GEV distribution. This is because the "difference between r-bias (t_3) and r-bias (g)" is less in case of GPA distribution. In the case of GNO and P-3 distributions, for sample lengths of 30 and above, r-std (t_3) is found to be less than r-std (g), resulting in a significantly lower value of r-RMSE (t_3), compared to r-RMSE (g). Furthermore, the difference in r-RMSE (t_3) and r-RMSE (g) for GNO is seen to be less than that of P-3 distribution. This is because r-std (t_3) is found to be somewhat higher than r-std (g), in case of GNO distribution.

The spread of the 10 000 sample estimates of L-skewness and conventional skewness is shown in Figs. 1 and 2 for two typical distributions namely, GEV and P-3, for six population skewnesses covering the range 0.5-5 and three sample sizes (30, 60 and 90). In Figs. 1 and 2, the dark line within the box represents the median, the top and the bottom of the box corresponding to the upper and the lower quartiles; and the upper arm contains markings of the upper 5 percentile and the maximum, while the lower arm extends to the minimum on which the lower 5 percentile is also marked. It is to be noted that the box plot comparisons are also shown in relative form for the sake of better understanding and appreciation. The line through 1.0 is the common relative reference used for comparison and that is nothing but the population skewness, with reference to itself. Figs. 1 and 2 show that the sampling distribution of L-skewness is normal, while that of conventional skewness is not. Furthermore, the trend of result comparison presented in the summary table (Table 10) concerning the sampling properties of L-skewness and conventional skewness is very much confirmed by the box plots shown in Figs. 1 and 2. For brevity, these plots are shown only for two typically selected distributions.

Typical results of the comparison of sampling properties between L-skewness and conventional skewness are presented in Table 11 for the extended range of population skewness 6–15 for the GEV distribution. These results indicate that as the skewness increases further, the superiority of L-moments over conventional moments is much more pronounced. This trend is found to be almost the same for all the other three 3-parameter distributions considered. For brevity, the entire results are not presented.

8. Application example

As an example, an application in the field of regional frequency analysis is presented herein, using annual maximum flow records collected from 98 catchments located within the central part of India. This application is presented with the view to bring out the efficacy of L-moments in the general identification of possible regional flood frequency distributions for a considered collection of number of

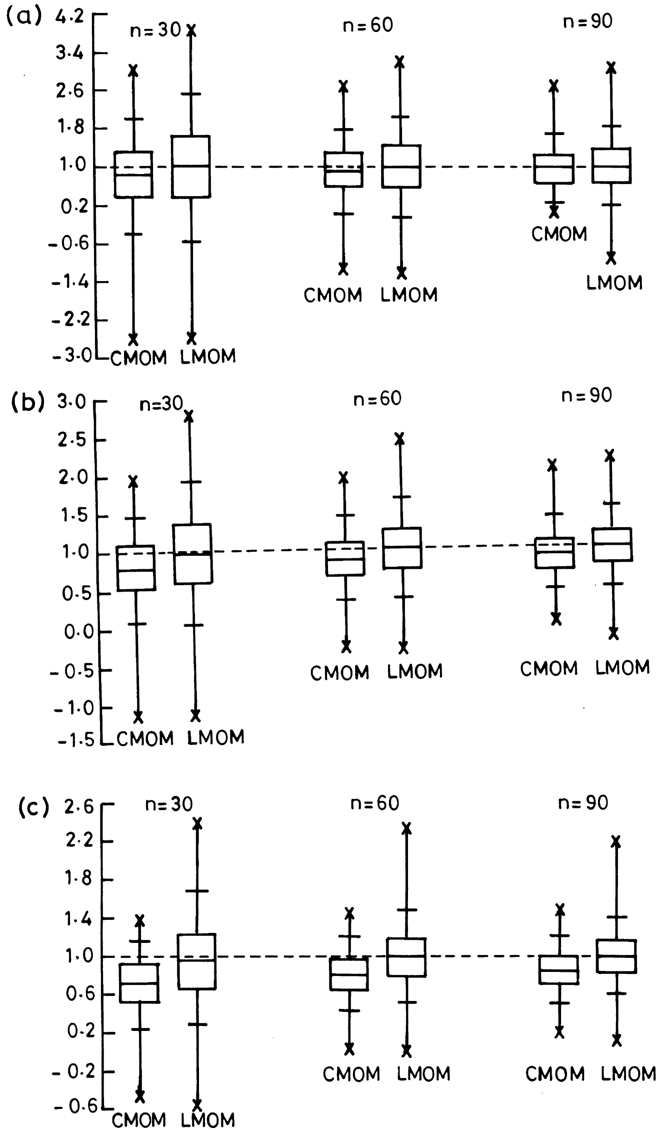

Fig. 1. Box plots showing the comparison of the spread of third moment ratios for: (a) gamma = 0.52; (b) gamma = 1.03; (c) gamma = 1.97; (d) gamma = 3.035; (e) gamma = 4.19; (f) gamma = 4.766. Distribution: GEV.

catchments within a large geographical area, wherein there may be a mix of sites with flood data having high as well as low Cv and skewness.

One of the main applications of L-moments is in the identification of the probability distribution of the observed phenomena using the L-moment ratio diagram. This is usually accomplished by comparing the sample estimates of L-moment ratios L-Cv, L-skewness, L-kurtosis with the population (theoretical) L-moment ratios for a range of assumed probability distributions. Normally, a plot between L-Cv and L-skewness is used to identify two-parameter distributions, while a plot between L-skewness and L-kurtosis is used to identify three-parameter distributions. Vogel and Wilson (1996) have pointed out that the evaluations of McMahon et al. (1992) and Finlayson and McMahon (1992) in connection with the exploration of probability distributions of annual maximum

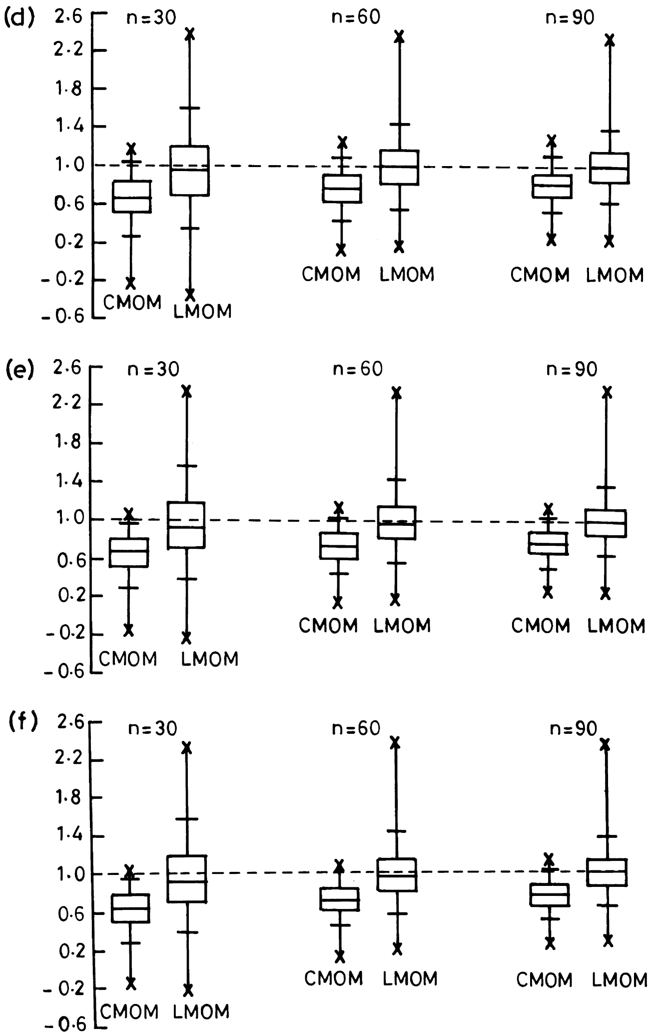

Fig. 1. (*continued*)

flood flows at 974 stations around the world using product moment ratio diagrams, are not definitive due to the significant bias displayed by the product moment ratios, especially for the highly skewed small samples.

L-moment ratio diagrams have been used by Wallis (1988); Pearson (1991); Pilon and Adamowski (1992); Karim and Chowdhury (1993); Vogel et al. (1993a, b); Vogel and Fennessey (1993); Rao and Hamed (1994); Onoz and Bayazit (1995) and Vogel and Wilson (1996) to assess the goodness of fit of various PD's to regional samples of flood flows. L-moment ratio diagrams have been useful in revising the regional frequency distributions suggested by earlier investigators [e.g.: Hosking and Wallis (1987) and Wallis (1988)]. Vogel and Wilson (1996) have documented that GEV, LN3 and LP3 distributions are adequate PD's to model the frequency of annual maximum flood flows in the continental United States. They have also suggested that the Pareto-based distribution introduced by Durrans (1994) would also be acceptable for modelling the flood flows throughout the continental United States. Recently, Madsen et al. (1997) have used L-moment

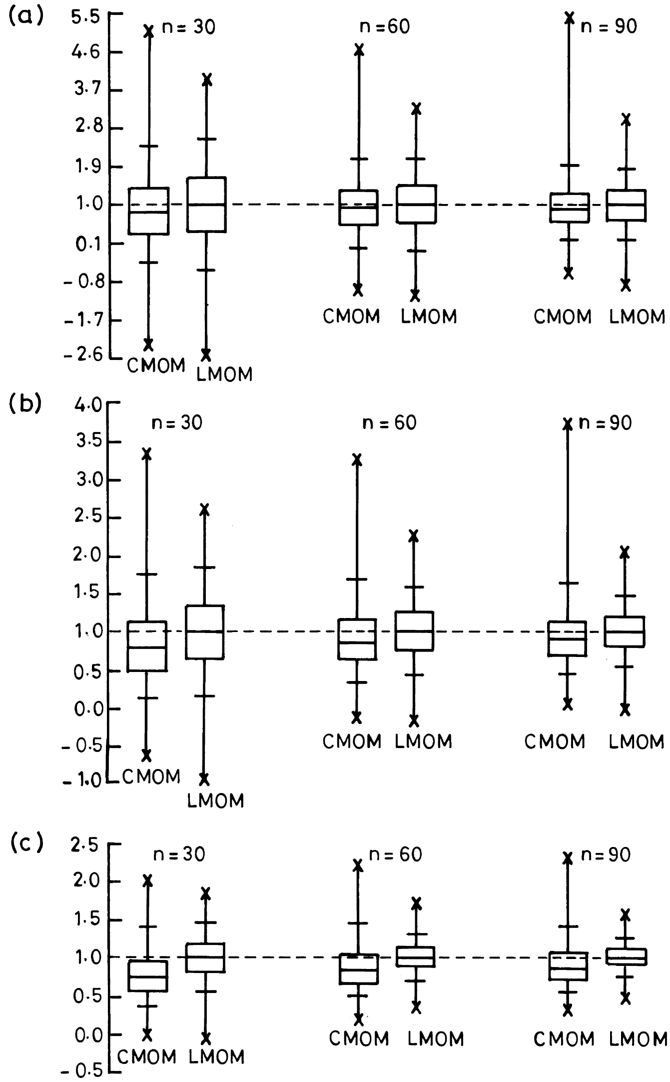

Fig. 2. Box plots showing the comparison of the spread of third moment ratios for: (a) gamma = 0.55; (b) gamma = 1.03; (c) gamma = 2.04; (d) gamma = 3.01; (e) gamma = 4.06; (f) gamma = 5.06. Distribution: Pearson-3.

ratio diagrams for the identification of distributions for both partial duration series (PDS) and annual max. series (AMS) of flood data. Some of the recent studies conducted in the area of regional flood frequency analysis for Indian catchments include RDSO (1991); NIH (1990–91; 1994–95a, b; 1995–96). These studies have attempted to fit regional flood frequency models to the subzones individually. However, none of these studies have used the L-moment ratio diagram for the identification of appropriate regional flood frequency distribution.

Hosking and Wallis (1993) list the four stages in a regional flood frequency analysis as: (i) screening of the data; (ii) identification of homogeneous regions; (iii) choice of a regional frequency distribution; and (iv) estimation of the regional frequency distribution. As the 98 catchments considered are spread over six hydrometeorologic subzones, representing different hydrology, a high degree of heterogeneity is to be

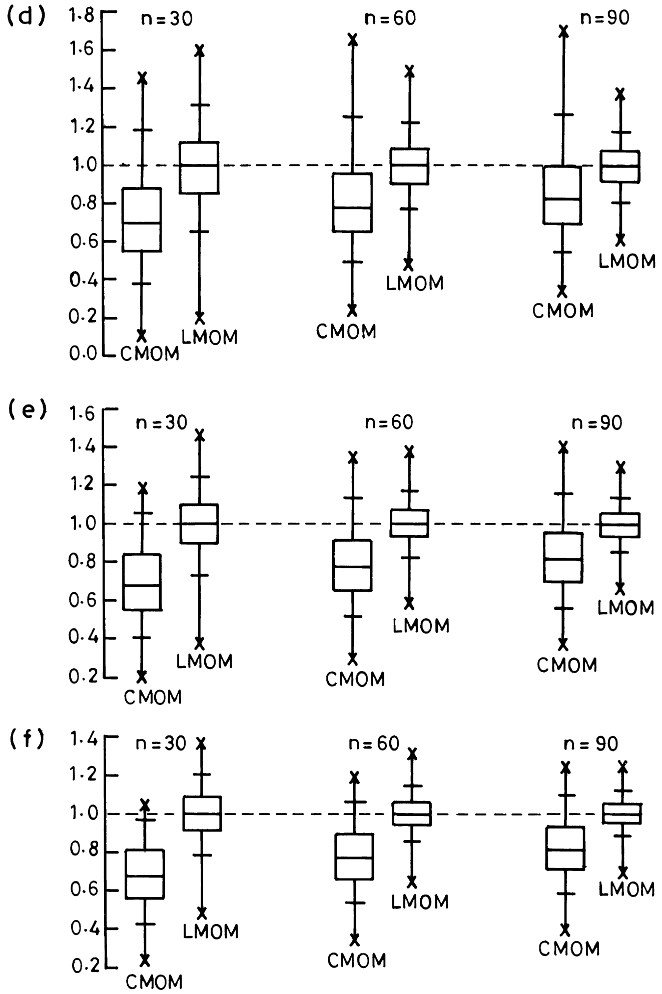

Fig. 2. (*continued*)

expected. However, since the idea of this application is to present a comparison of L-moment ratio diagrams (based on L-moment ratios L-Cv, L-skewness and L-kurtosis), with the Product moment ratio diagrams (based on conventional moment ratios Cv, skewness and kurtosis), in selecting suitable probability distributions for the considered region, it is not necessary to group the region into homogeneous basins. Identification of homogeneous regions is important, only if the final goal is to predict the at-site flood quantiles, based on the estimated regional frequency distribution (Vogel and Wilson, 1996).

The region considered for the application is located in the central part of the Indian subcontinent and it extends from 18° N to 25° N latitude and from 70° E to 87° E longitude (Fig. 3). Six hydrometeorologic subzones (3a–3f) are taken together as a single region covering most parts of the six river basins Mahi, Sabarmati, Narmada, Tapi and Godavari and Mahanadi. The first four rivers are west flowing and fall out into the Arabian sea on the western side, while the latter two rivers are east flowing and fall out into the Bay of Bengal on the eastern side. This region is geographically bounded by the North Central Highlands in the north, the South Deccan Plateau in the south, the Eastern Coastal plains in the east and the

Table 11
Comparison of sampling properties of L-skewness with conventional skewness (for skewness $>$ 5.0)-GEV distribution

n	Skew	L-skew	r-bias (t_3)[a]	r-bias (g)[b,e]	r-RMSE (t_3)[c]	r-RMSE (g)[d,e]
20	6.531	0.350	0.0981	0.4480	0.4268	0.4937
20	8.003	0.360	0.1012	0.4551	0.4207	0.4973
20	10.345	0.370	0.1044	0.4674	0.4151	0.5063
20	14.690	0.380	0.1076	0.4814	0.4097	0.5174
30	6.531	0.350	0.0731	0.3919	0.3538	0.4329
30	8.003	0.360	0.0756	0.3999	0.3496	0.4376
30	10.345	0.370	0.0783	0.4136	0.3457	0.4481
30	14.690	0.380	0.0810	0.4290	0.3421	0.4607
60	6.531	0.350	0.0411	0.3122	0.2620	0.3493
60	8.003	0.360	0.0428	0.3214	0.2600	0.3552
60	10.345	0.370	0.0447	0.3369	0.2581	0.3675
60	14.690	0.380	0.0466	0.3544	0.2564	0.3821
90	6.531	0.350	0.0282	0.2723	0.2188	0.3073
90	8.003	0.360	0.0295	0.2821	0.2176	0.3138
90	10.345	0.370	0.0310	0.2985	0.2166	0.3269
90	14.690	0.380	0.0324	0.3170	0.2157	0.3425

[a] Here r-bias (t_3)-Relative bias in L-skewness.
[b] Here r-bias (g)-Relative bias in skewness.
[c] Here r-RMSE (t_3)-Relative RMSE in L-skewness.
[d] Here r-RMSE (g)-Relative RMSE in skewness.
[e] Converted to equivalent values in the L-plane.

Western Coastal plains in the west. Out of the 98 catchments considered in the region, 10 are located in the Mahi and Sabarmati subzone (3a), 19 lie in the Lower Narmada and Tapi subzone (3b), 15 are in the Upper Narmada and Tapi subzone (3c), 23 belong to the Mahanadi subzone (3d), 12 are contained in the Upper Godavari subzone (3e) and 19 are within the Lower Godavari subzone (3f) (Fig. 3).

The average record length of the annual maximum floods at the 98 sites of this region is 21.65 years and the average drainage area for the 98 sites is 301 km^2. The theoretical relationship between L-skewness and L-kurtosis for various three-parameter distributions is constructed using the polynomial approximations given in Hosking (1991). Five candidate distributions are considered, namely, Generalized Logistic, Generalized Extreme Value, Generalized Pareto, Generalized Normal and Pearson-3 distribution. Fig. 4 compares the observed and the theoretical relations between L-skewness and L-kurtosis for the annual maximum flood flows at the 98 sites. It is observed from the L-moment ratio diagram (Fig. 4) that out of the five candidate distributions, Generalized Pareto distribution seems to fit the observed moments well. It almost bisects the sample L-moment ratios into two halves. All the other distributions seem to fit only a part of the observed moments. Fig. 5 compares the observed and the theoretical relations between skewness and kurtosis for the annual maximum flood flows at the 98 sites considered. Again, out of the five distributions, Generalized Pareto distribution seems to give a reasonable fit to the sample moments from the 98 sites. Hence, the Generalized Pareto distribution may be suggested for the annual maximum flood flows for the region considered.

However, it may be noted from the product moment ratio diagram (Fig. 5) of this region that for sites with lower sample skewness, the sample product moment ratios are close to the theoretical curve of Generalized Pareto distribution, and are scattered on either side. This is because for $\gamma < 1.8$, the samples from the Generalized Pareto distribution do not exhibit much bias even for record lengths 20–30, while the variance is considerable (refer Table 8). In contrast, for sites with higher skewness, the sample product moment ratios are seen to underestimate their theoretical counterparts even for the suggested GPA distribution and for all the remaining four distributions as well. This is

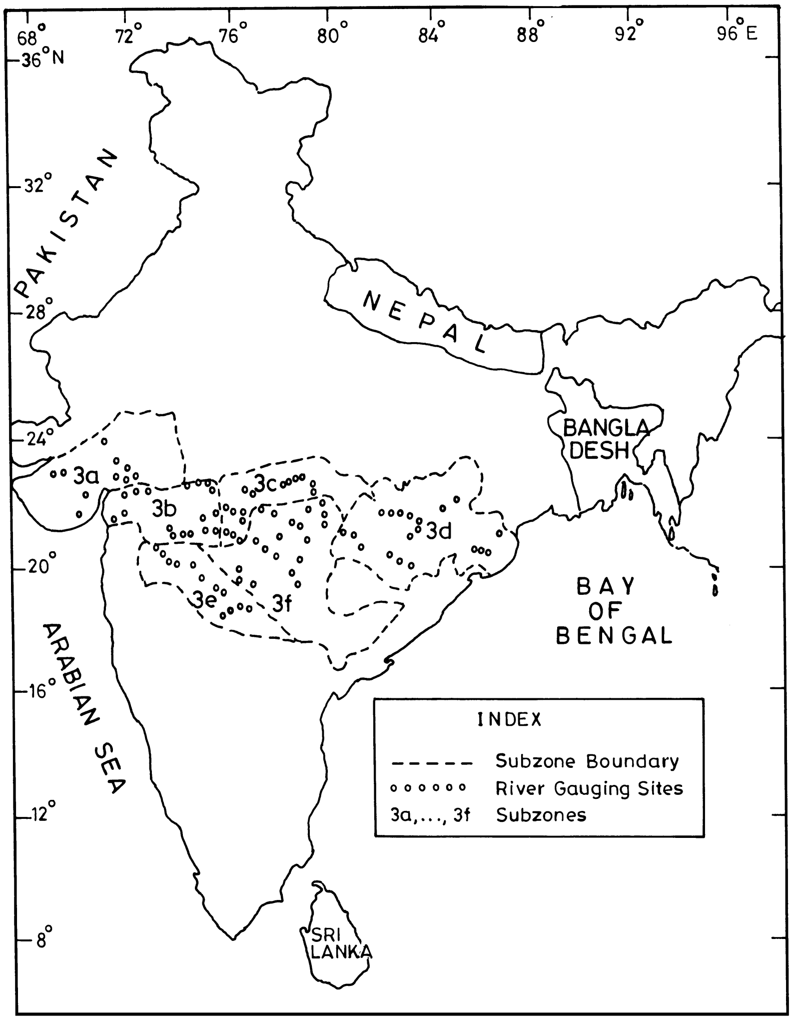

Fig. 3. Location of the 98 river gauging sites — Zone 3 of India.

because of the significant downward bias exhibited by product moment ratios at higher skewness (Tables 6–9). However, the estimators of L-moments seem to estimate the theoretical moment ratios reasonably well, even under high skewness, as seen from Fig. 4. The simulation results presented in Tables 6–9 validate the previous statement, as the r-RMSE of 't3' is considerably less than that of r-RMSE of 'g' for population skewness above 1.8, for the sample size range encountered.

As observed in this application, any realistic region may consist of a mix of sites with flood data having high as well as low skewness. Sometimes, the record lengths at certain sites also may be very limited. In such cases, product moment ratio diagrams and product moment ratio based estimators cannot provide a consistent and dependable description for the entire sites in the region, while L-moment ratio diagrams and L-moment based estimators can provide a reasonable and consistent description for the entire sites in the region, including the outlying ones. Thus, it is clearly illustrated from this application that L-moment ratio diagrams are much more consistent and dependable compared to the conventional product moment ratio diagrams in the identification of the regional flood frequency distributions of large geographical regions.

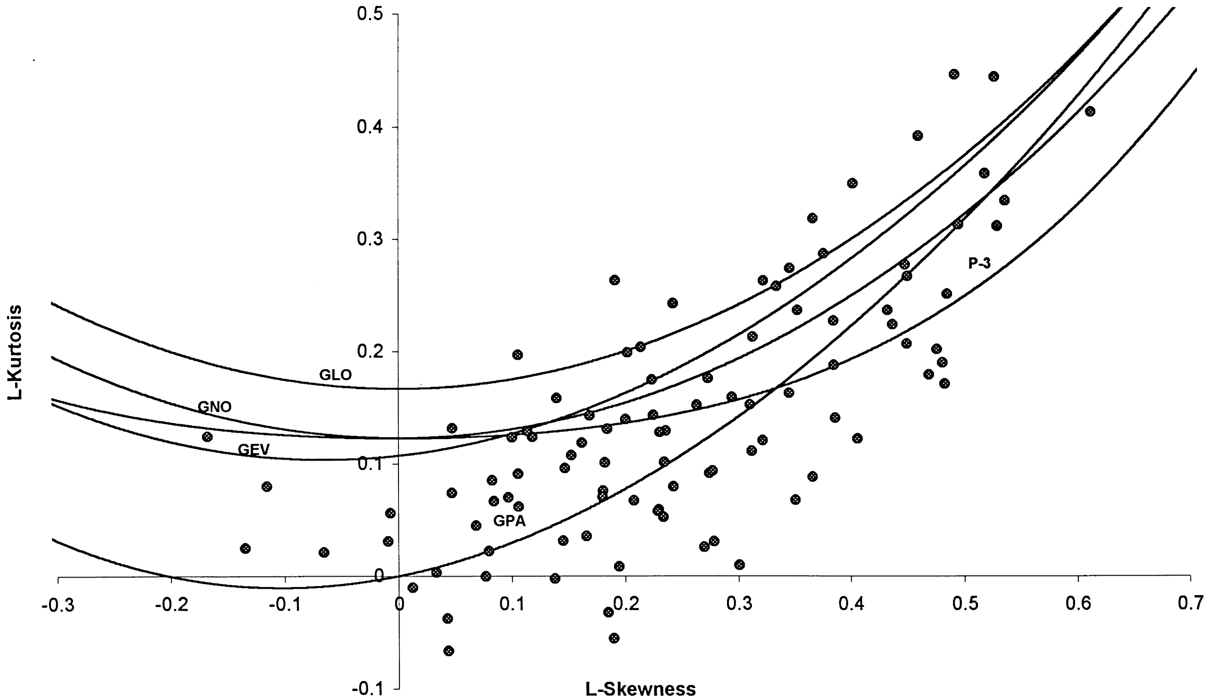

Fig. 4. L-moment ratio diagram for annual maximum streamflows for 98 sites — Zone 3 of India.

9. Summary and conclusions

Based on the detailed Monte-Carlo simulation results, regression equations have been fitted for variance in L-standard deviation; and bias and variance in L-Skewness for two of the commonly used distributions in hydrology and water resources applications, namely: generalised Normal and Pearson-3 distributions. Moreover, the sampling properties namely, RMSE in second moment; and bias and RMSE in third moment ratio of L-Moments have been compared with those of Conventional Moments in a relative form in the same plane (L-plane) and the results of the comparison have been presented for the four distributions, namely, generalised Normal, generalised Extreme Value, generalised Pareto and Pearson-3 distributions.

There is no significant difference in the relative RMSE in second moment between conventional moments and L-moments when the sample size is 10, whereas with increase in sample size, the performance of L-moments is found to improve considerably over conventional moments, especially for higher skewness. The bias in L-skewness is found to be very small upto a skewness of about 1.0, even for small samples. In case of higher skewness, for a reasonable sample size of 30, the bias in L-skewness is found to be very less. On the contrary, the conventional skewness is found to be significantly biased, even for a low skewness of 0.5 and a reasonable sample size of 30. The performance evaluation in terms of “Relative-RMSE in third moment ratio”, reveals that conventional moments are preferable at lower skewness, particularly for smaller samples, while L-moments are preferable at higher skewness, for all sample sizes.

Even for lower skewness values, as the sample size increases, the performance of L-moments improves considerably and the difference in relative RMSE between L-skewness and conventional skewness reduces phenomenally. However, at higher skewness, the improvement in the performance of conventional moments with increase in sample size is comparatively less and hence the superiority in performance in terms of “relative RMSE in third moment ratio” is retained by

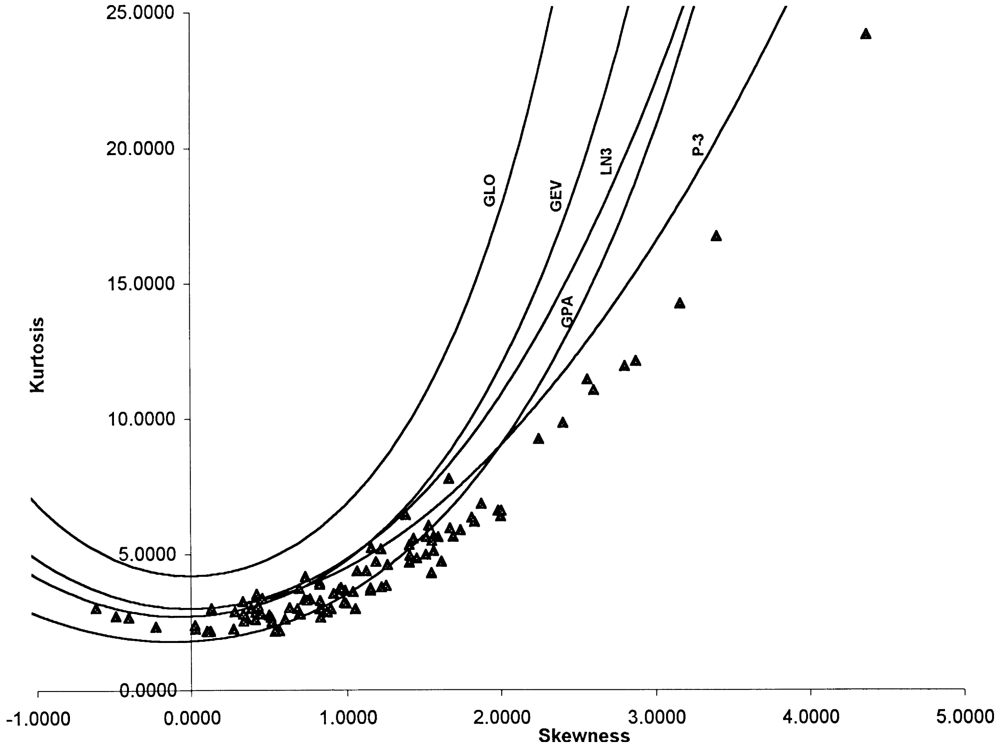

Fig. 5. Product moment ratio diagram for annual maximum streamflow for 98 sites — Zone 3 of India.

L-moments at higher skewness, even when the sample size is increased to 90.

The results regarding the sampling properties of moment ratios presented in this article will be useful in understanding and interpreting the moment ratio diagrams in the context of the selection of regional frequency distributions for hydrologic studies. This is illustrated through an application example that seeks to obtain an appropriate regional flood frequency distribution for the central region of India.

As an extension of this research work, the sampling properties of L-Cv are being investigated by the authors for a number of distributions used in hydrology and a comparison of the same with that of Cv is also being attempted.

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