Chapter 9

STRESSES AND DISPLACEMENTS IN A NON-HOMOGENEOUS ELASTIC MASS

9.1 Semi-Infinite Mass with Linear Variation of Modulus

9.1.1 UNIFORM STRIP LOADING

This problem has been solved by Gibson (1967) who considered an elastic half-space where Poisson's ratio remains constant but the shear modulus G increases linearly with depth as follows (Fig. 9.1):

$$G(z) = G(0) + mz \qquad \dots (9.1)$$

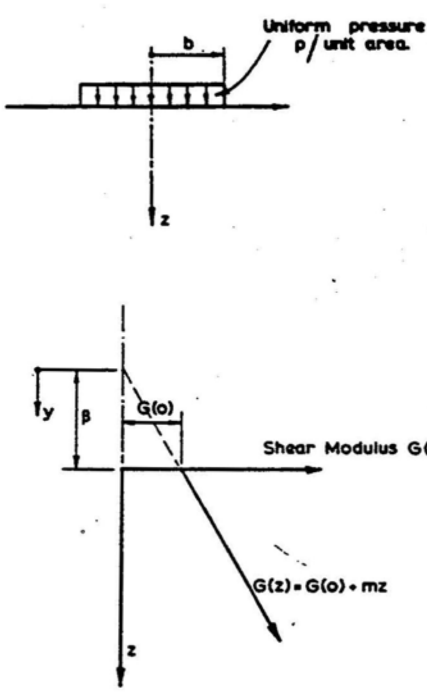


FIG.9.1

Two limiting cases may be recovered from the general expressions for $\nu=0.5$:

(i) $m=0$ $(\beta=\infty)$ i.e. constant $G=G(0)$ with depth. This is the classical homogeneous case.

(ii) $G(0)=0$ $(\beta=0)$. This is the case of a linear increase in G with depth, starting from zero at the surface.

Considering the vertical displacement ρ_z for each case, it is found that for case (i), the actual displacement is infinite (see Section 3.1), but the difference between the displacement of a point and the central surface displacement is finite, and equal to

$$\rho_z(0,0)-\rho_z(x,z) = \frac{pb}{2\pi G(0)}\Big\{\tfrac{1}{2}\Big(1+\frac{x}{b}\Big)\ \ln\Big[\frac{z^2+(b+x)^2}{b^2}\Big] + \tfrac{1}{2}\Big(1-\frac{x}{b}\Big)\ \ln\Big[\frac{z^2+(b-x)^2}{b^2}\Big] + \frac{z}{b}\tan^{-1}\Big(\frac{b+x}{z}\Big) + \frac{z}{b}\tan^{-1}\Big(\frac{b-x}{z}\Big)\Big\} \qquad \dots (9.2)$$

Values of $(\rho_z(0,0)-\rho_z(x,z))$ are plotted in Fig.9.2(a).

For case (ii), the vertical displacement is now finite, and given by

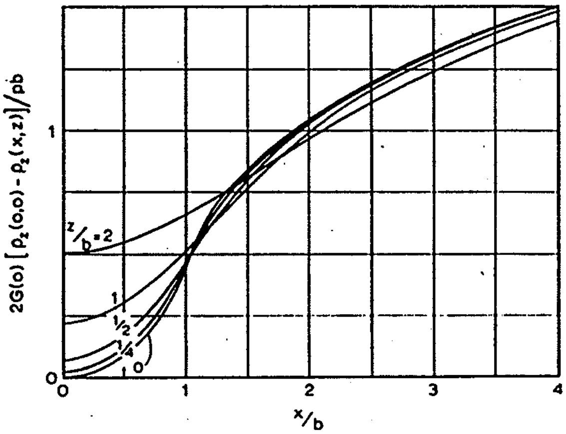
$$\rho_z = \frac{p}{2\pi m}\Big[\tan^{-1}\Big(\frac{b+x}{z}\Big) + \tan^{-1}\Big(\frac{b-x}{z}\Big)\Big] \qquad \dots (9.3)$$

Values of ρ_z for this case are plotted in Fig. 9.2(b).

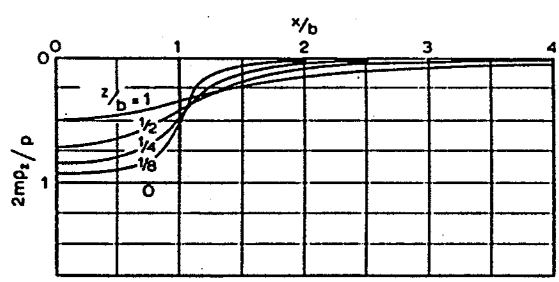
It is noteworthy that the stresses are *identical* for both cases (i) and (ii) i.e. non-homogeneity has no influence on the stress distribution. This result suggests that stresses for finite values of β may not differ appreciably from the values for the limiting cases, provided $\nu=0.5$.

An important conclusion reached from a study of case (ii) is that a material whose modulus varies linearly with depth from zero at the surface behaves as a "Winkler" material i.e. the vertical displacement at any point on the surface is directly proportional to the intensity of vertical stress at that point. This may clearly be seen from the curve for $z/b=0$ in Fig.9.2(b). The coefficient of subgrade reaction k_s in the Winkler material is related to m as $k_s=2m$.

It should be noted that the above conclusion is exact only for $\nu=0.5$.



(a) Relative displacements of strip on uniform mass. ν-0.5. G=G(O).



(b) Displacement of strip on mass with linearly increasing G. ν=0.5. G(O)=O.

FIG.9.2 (Gibson, 1967)

9.1.2 UNIFORM LOADING OVER CIRCULAR AREA

Profiles of vertical surface displacement in terms of the value at the centre have been obtained by Brown and Gibson (1972) for three values of ν and are shown in Figs. 9.3 to 9.5. In these figures, r is the radial distance from the centre, a is the radius and β is as defined in Fig. 9.1.

The variation of central surface vertical displacement $\rho_z(r=0)$ with β and ν is shown in Fig. 9.6.

The conclusion regarding the identical behaviour of an incompressible $(\nu=0.5)$ mass whose modulus varies linearly with depth from zero at the surface, and a Winkler medium, remains valid for this type of loading and indeed, for any type of surface loading.

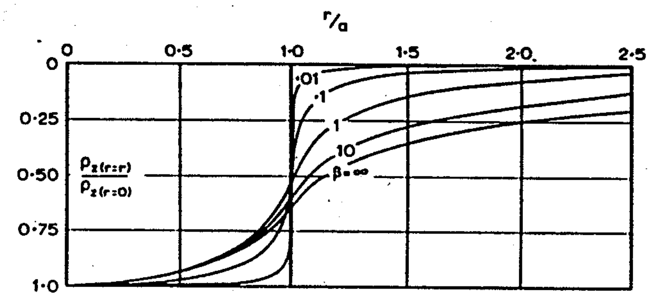


FIG.9.3 Surface displacement profiles for ν=0.5 (Brown and Gibson, 1972).

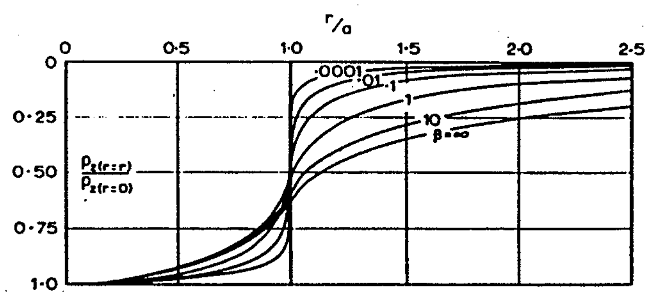


FIG.9.4 Surface displacement profiles for ν=1/3 (Brown and Gibson, 1972).

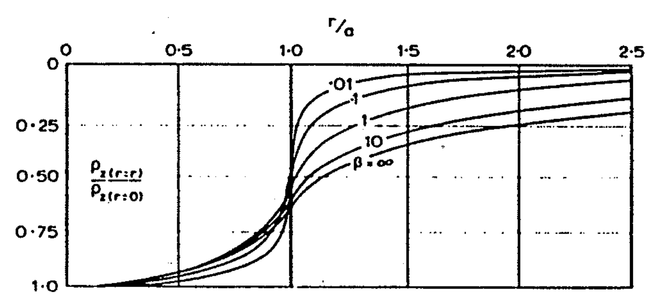


FIG.9.5 Surface displacement profiles for ν=0 (Brown and Gibson, 1972).

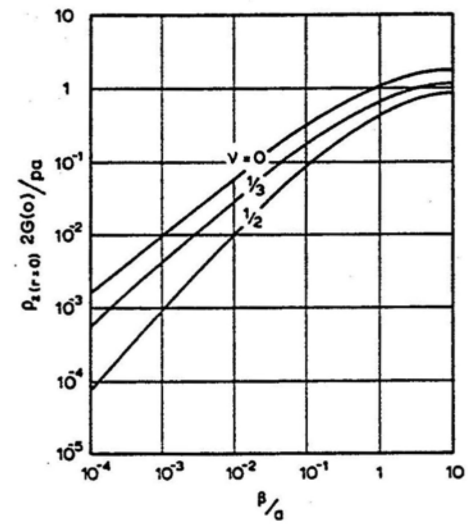


FIG.9.6 Central surface displacement of circular area. (Brown and Gibson, 1972).

p = applied pressure
a = radius

9.2 Generalized Boussinesq Theory for Non-Homogeneous Semi-Infinite Mass

9.2.1 VERTICAL POINT LOADING ON SURFACE

Holl (1940) developed a general form of Boussinesq's classical equations, based on earlier solutions of Griffith (1929) and Frohlich (1933). The solutions for a vertical point loading (Fig. 2.2) are:

$$\sigma_z = \frac{nPz^n}{2\pi R^{n+2}} \quad \ldots (9.4a)$$

$$\sigma_r = \frac{nPz^{n-2}r^2}{2\pi R^{n+2}} \quad \ldots (9.4b)$$

$$\sigma_\theta = 0. \quad \ldots (9.4c)$$

$$\tau_{rz} = \frac{nPz^{n-1}r}{2\pi R^{n+2}} \quad \ldots (9.4d)$$

The above solutions are valid for $n>2$ and satisfy equilibrium and compatibility requirements for the following restricted class of material:

$$E = E_o z^\lambda \quad \ldots (9.5)$$

where E_o = modulus at unit depth
$\lambda = n-3 = \frac{1}{\nu} - 2.$

When $n=3$, $\lambda=0$ and $\nu=0.5$, the above solutions reduce to the classical Boussinesq solutions for $\nu=0.5$. When $n=4$, $\lambda=1$ and $\nu=1/3$, the modulus E varies linearly with depth. Thus, this case corresponds to that considered by Gibson (1967) (Section 9.1) except that Gibson considered $\nu=0.5$.

Provided the above restrictions on ν and modulus variation are observed the generalized Boussinesq solution may be used to study the stress distribution in a non-homogeneous mass for all types of surface loading.

9.2.2 HORIZONTAL POINT LOADING ON SURFACE

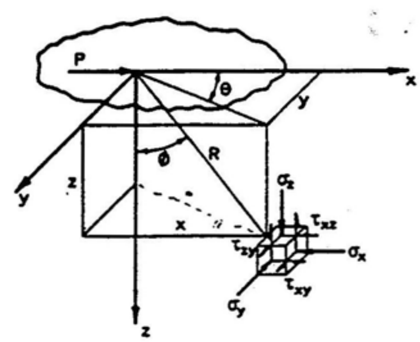


FIG.9.7

Holl (1940) gives the following solutions (refer Fig. 9.7):

$$\sigma_z = \frac{n(n-2)P}{2\pi z^2} \cos^{n+1}\phi \sin\phi \cos\theta \quad \ldots (9.6a)$$

$$\sigma_x = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos^3\theta \quad \ldots (9.6b)$$

$$\sigma_y = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos\theta \sin^2\theta \ . \ (9.6c)$$

$$\tau_{yz} = \frac{n(n-2)P}{2\pi z^2} \cos^{n}\phi \sin^2\phi \sin\theta \cos\theta \quad \ldots (9.6d)$$

$$\tau_{xy} = \frac{n(n-2)P}{2\pi z^2} \cos^{n}\phi \sin^2\phi \cos^2\theta \quad \ldots (9.6e)$$

$$\tau_{xz} = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \sin\theta \cos^2\theta \ . \ (9.6f)$$

Note that the same restrictions on the relationship between modulus variation and ν apply in this case as in the case of a vertical point load (section 9.2.1).

9.2.3 LINE LOADING ON SURFACE (Fig. 9.8)

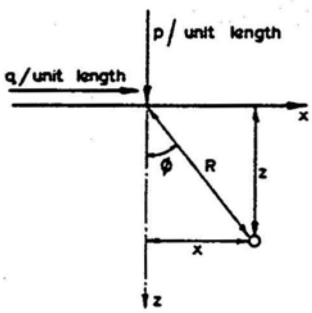


FIG.9.8

(a) Vertical Loading

$$\sigma_z = \frac{p}{z} K \cos^{n+1}\phi \qquad \text{... (9.7a)}$$

$$\sigma_x = \frac{p}{z} K \cos^{n-1}\phi \sin^2\phi \qquad \text{... (9.7b)}$$

$$\tau_{xz} = \frac{p}{z} K \cos^{n}\phi \sin\phi \qquad \text{... (9.7c)}$$

Values of K are tabulated in Table 9.1.

(b) Horizontal Loading

$$\sigma_z = (n-2)K \frac{q}{z} \cos^{n}\phi \sin\phi \qquad \text{... (9.8a)}$$

$$\sigma_x = (n-2)K \frac{q}{z} \cos^{n-2}\phi \sin^3\phi \qquad \text{... (9.8b)}$$

$$\tau_{xz} = (n-2)K \frac{q}{z} \cos^{n-1}\phi \sin^2\phi \qquad \text{... (9.8c)}$$

Values of K are tabulated in Table 9.1 (same as for vertical loading).

TABLE 9.1

K versus n

(Holl, 1940)

n	3	4	5	6	7	8
K	$\frac{2}{\pi}$	$\frac{3}{4}$	$\frac{8}{3\pi}$	$\frac{15}{16}$	$\frac{16}{5\pi}$	$\frac{35}{32}$

9.2.4 UNIFORM VERTICAL LOADING OVER CIRCULAR AREA

Beneath the centre of a circle of radius a, loaded with a uniform surface stress p per unit area,

$$\sigma_z = p[1 - \cos^n\phi_0] \qquad \text{... (9.9a)}$$

$$\sigma_r = \frac{p}{2}\left[\frac{n}{n-2}(1 - \cos^{n-2}\phi_0) - (1 - \cos^n\phi_0)\right] \qquad \text{... (9.9b)}$$

where $\phi_0 = \tan^{-1}\frac{a}{z}$

Values of σ_z are tabulated in Table 9.2 for $n=4$ and 5. Holl (1940) also gives expressions for σ_z and σ_r due to parabolic vertical loading.

TABLE 9.2

GENERALISED BOUSSINESQ PROBLEM

VALUES OF σ_z/p BENEATH CENTRE

Uniform Vertical Loading

(After Harr, 1966)

n	$\frac{z}{a}$	Circle Radius a	Rectangle $m=1$	Rectangle $m=2$	Rectangle $m=3$	Rectangle $m=10$	Infinite Strip $m=\infty$
4	0	1	1	1	1	1	1
	0.25	0.996	0.942	0.950	0.957	0.970	0.984
	0.5	0.960	0.792	0.824	0.852	0.876	0.884
	1	0.750	0.426	0.547	0.581	0.603	0.625
	1.5	0.518	0.255	0.372	0.404	0.437	0.457
	2	0.360	0.142	0.230	0.281	0.316	0.357
	3	0.109	0.074	0.135	0.175	0.214	0.245
	5	0.077	0.025	0.048	0.063	0.125	0.150
5	0	1	1	1	1	1	1
	0.25	0.998	0.984	0.987	0.988	0.991	0.993
	0.5	0.982	0.859	0.830	0.902	0.922	0.925
	1	0.817	0.508	0.625	0.657	0.676	0.686
	1.5	0.591	0.298	0.404	0.443	0.480	0.510
	2	0.429	0.164	0.249	0.294	0.333	0.411
	3	0.229	0.089	0.150	0.192	0.243	0.282
	5	0.096	0.032	0.062	0.088	0.141	0.162

9.2.5 UNIFORM VERTICAL LOADING OVER RECTANGULAR AREA (Fig. 9.9)

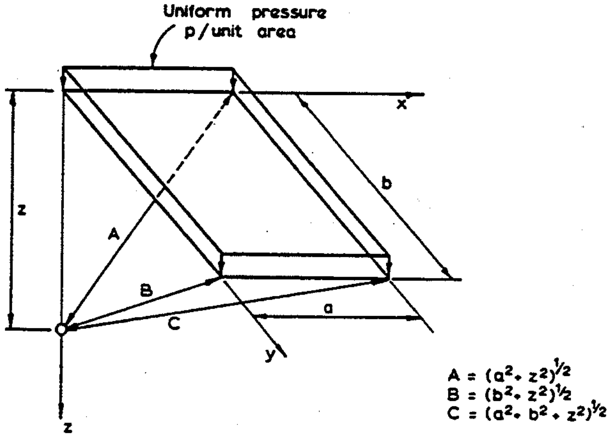


FIG.9.9

Beneath the corner of the rectangle, for $n=4$,

$$\sigma_z = \frac{p}{2\pi}\left[\frac{a}{A}\left(1+\frac{z^2}{2A^2}\right)\arctan\frac{b}{A} + \frac{b}{B}\left(1+\frac{z^2}{2B^2}\right)\arctan\frac{a}{B} + \frac{abz^2}{2C^2}\left(\frac{1}{A^2}+\frac{1}{B^2}\right)\right] \quad \ldots (9.10a)$$

$$\sigma_x = \frac{p}{4\pi}\left[\frac{a^3}{A^3}\arctan\frac{b}{A} + \frac{b}{B}\arctan\frac{a}{B} - \frac{abz^2}{A^2C^2}\right] \quad \ldots (9.10b)$$

$$\tau_{xz} = \frac{p}{4\pi}\left[\arctan\frac{b}{z} - \frac{z^3}{A^3}\arctan\frac{b}{A} + bz\left(\frac{1}{B^2}-\frac{z^2}{A^2C^2}\right)\right] \quad \ldots (9.10c)$$

$$\tau_{xy} = \frac{p}{4\pi}\left[\frac{(a^2+b^2)^2}{z^2C^2} - \frac{1}{z^2}\left(\frac{a^4}{A^2}+\frac{b^4}{B^2}\right)\right] \quad \ldots (9.10d)$$

Holl (1940) also quotes expressions for n values of 3 (homogeneous mass, see Chapter 3), $5, 6, 7$ and 8.

Values of σ_z for $n=4$ and 5, given by Harr (1966), are reproduced in Table 9.2 for various values of $m=b/a$.

9.2.6 UNIFORM HORIZONTAL LOADING OVER RECTANGULAR AREA (Fig. 9.10)

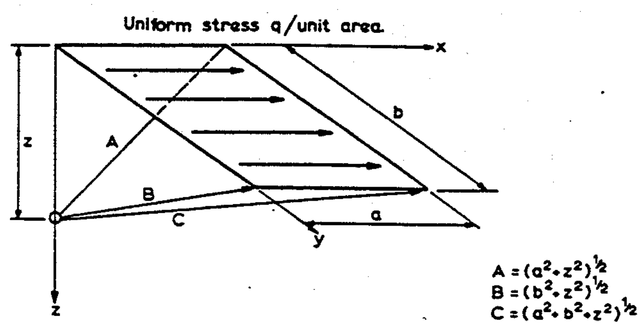


FIG.9.10

For $n=4$,

$$\sigma_x = \frac{q}{\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\left(1+\frac{a^2}{2A^2}\right)\arctan\frac{b}{A} - \frac{a^2bz}{2A^2C^2}\right] \quad \ldots (9.11a)$$

$$\sigma_y = \frac{q}{2\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\arctan\frac{b}{A} - bz\left(\frac{1}{B^2}-\frac{1}{C^2}\right)\right] \quad \ldots (9.11b)$$

$$\tau_{xy} = \frac{q}{2\pi}\left[\arctan\frac{a}{z} - \frac{z}{B}\arctan\frac{a}{B} + \frac{a}{z}\left(\frac{a^2}{A^2}-\frac{a^2+b^2}{C^2}\right)\right] \quad \ldots (9.11c)$$

Holl (1940) also quotes solutions for $n=3$ (homogenous case, see Chapter 3) and $n=5$.

The values of τ_{xz}, τ_{yz} and σ_z for a horizontal loading correspond to the values of σ_x, τ_{xy} and τ_{xz} for a vertical loading, multiplied by the factor $(n-2)$.

9.3 Finite Layer with Linear Variation of Modulus (Fig. 9.11)

This problem has been considered by Gibson, Brown and Andrews (1971). Profiles of vertical surface displacement due to uniform strip loading are shown in Fig. 9.12 and due to uniform circular loading in Fig. 9.13. In both figures, $G(0)=0$ and $\nu=0.5$. In this case the vertical displacement of the loaded area is strictly uniform only when h/b or $h/a=\infty$; as the layer thickness decreases, the non-uniformity of settlement, inside and outside the loaded area, increases.

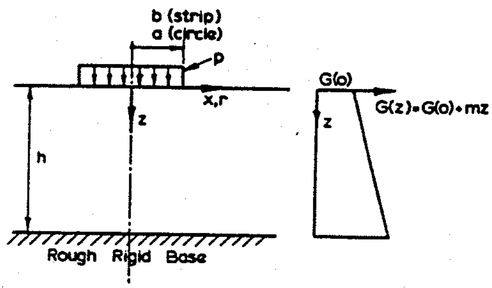


FIG.9.11

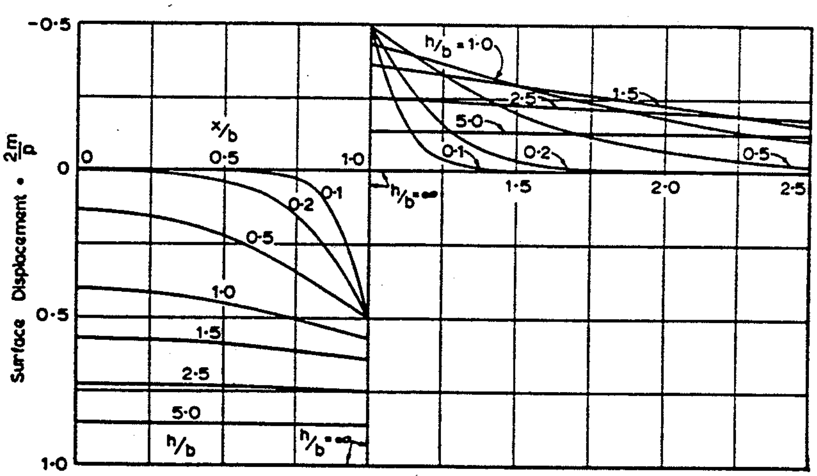


FIG.9.12 Surface displacement profiles due to uniform strip loading (Gibson et at, 1971).

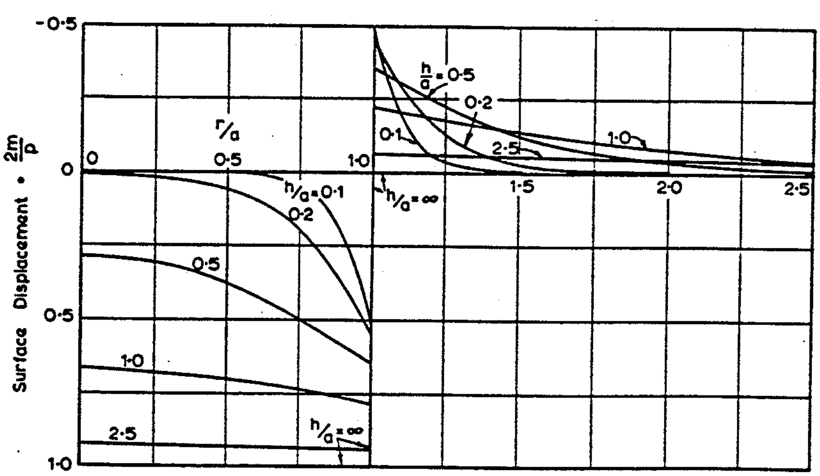


FIG.9.13 Surface displacement profiles due to uniform circular loading (Gibson et al, 1971).